Virtual Physics

Numerical Integration of Newton's Second Law Made Easy

Doug Harper

Department of Physics and Astronomy
Western Kentucky University

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Motivation

• Our typical approach to dynamics is to solve problems analytically.

• Often this requires that we make some fairly drastic assumptions.

• Textbook problems are often quite different from “Real World” problems.
Analytical Approach to Dynamics Problems
(Time Dependent Forces)

- For a particle of mass $m$ moving in one dimension, determine the net force $F_{\text{net,x}}$ acting on the particle by summing all the forces.

- Determine the acceleration using Newton’s Second Law

$$a_x(t) = \frac{F_{\text{net,x}}(t)}{m}$$

- Use this acceleration to determine the velocity from the relationship

$$a_x = \frac{dv_x}{dt} = \dot{v}_x \quad v_x(t) = \int a_x(t)dt$$

- Use this velocity to determine the position from the relationship

$$v_x = \frac{dx}{dt} = \dot{x} \quad x(t) = \int v(t)dt$$
More General Approach

• Forces may depend on position and/or velocity coordinates as well as time.

\[ F_{net} = F_{net}(x, v_x, t) = F_{net}(x, \dot{x}, t) \]

• Application of Newton’s Second Law results in a second-order differential equation.

\[ F_{net}(x, \dot{x}, t) = m \ddot{x} \]
\[ f(x, \dot{x}, \ddot{x}, t) = 0 \]
Numerical Integration by the Euler Method

• Express the net force on the particle as a function of position and velocity coordinates. One second order differential equation becomes two first order differential equations.

\[ F_{net}(x, v_x, t) = m \dot{v}_x \]

\[ \dot{v}_x = \frac{F_{net}(x, v_x, t)}{m} \quad \dot{x} = v_x \]

• Approximate derivatives as ratios of finite differentials.

• Iteratively, …
  – at a given time, determine acceleration from the net force;
  – calculate new velocity at that instant using the acceleration;
  – calculate new position at that instant using the velocity;
  – increment the time by \( \Delta t \) and repeat.
Velocity Determination from Euler Method

\[ a_x = \frac{dv_x}{dt} \approx \frac{\Delta v_x}{\Delta t} = \frac{v_x(t + \Delta t) - v_x(t)}{\Delta t} \]

\[ v_x(t + \Delta t) \approx v_x(t) + a_x(t) \Delta t \]

- average acceleration over time \( \Delta t \).
- velocity (x-component) at start of time interval.
- velocity (x-component) at end of time interval.
Position Determination from Euler Method

\[ v_x = \frac{dx}{dt} \approx \frac{\Delta x}{\Delta t} = \frac{x(t + \Delta t) - x(t)}{\Delta t} \]

\[ x(t + \Delta t) \approx x(t) + v_x(t) \Delta t \]

- velocity (x-component) at start of interval.
- position (x-component) at start of time interval.
- position (x-component) at end of time interval.
The Euler Method
Applied to 1-D Dynamics Problems

<table>
<thead>
<tr>
<th>Step</th>
<th>Time</th>
<th>Position</th>
<th>Velocity</th>
<th>Acceleration</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$t_0$</td>
<td>$x_0$</td>
<td>$v_0$</td>
<td>$a_0 = F(x_0,v_0,t_0)/m$</td>
</tr>
<tr>
<td>1</td>
<td>$t_1 = t_0 + \Delta t$</td>
<td>$x_1 = x_0 + v_0 \Delta t$</td>
<td>$v_1 = v_0 + a_0 \Delta t$</td>
<td>$a_1 = F(x_1,v_1,t_1)/m$</td>
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<tr>
<td>2</td>
<td>$t_2 = t_1 + \Delta t$</td>
<td>$x_2 = x_1 + v_1 \Delta t$</td>
<td>$v_2 = v_1 + a_1 \Delta t$</td>
<td>$a_2 = F(x_2,v_2,t_2)/m$</td>
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<tr>
<td>3</td>
<td>$t_3 = t_2 + \Delta t$</td>
<td>$x_3 = x_2 + v_2 \Delta t$</td>
<td>$v_3 = v_2 + a_2 \Delta t$</td>
<td>$a_3 = F(x_3,v_3,t_3)/m$</td>
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<tr>
<td>$n$</td>
<td>$t_n = t_{n-1} + \Delta t$</td>
<td>$x_n = x_{n-1} + v_{n-1} \Delta t$</td>
<td>$v_n = v_{n-1} + a_{n-1} \Delta t$</td>
<td>$a_n = F(x_n,v_n,t_n)/m$</td>
</tr>
</tbody>
</table>
Simple Harmonic Motion

\[ F_{\text{net},x} = m \frac{dv_x}{dt} \]

\[ F_{\text{spring}} = m\dot{v}_x \]

\[-k x = m\dot{v}_x \]

\[ \dot{v}_x = -(k/m)x \]

\[ \dot{x} = v_x \]
Virtual Physics

Differential Equations

<table>
<thead>
<tr>
<th>Variable Description</th>
<th>Symbol</th>
<th>Differential Equation</th>
<th>Initial Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Velocity</td>
<td>$v$</td>
<td>$v'(t) = -(k/m)x$</td>
<td>$v(0.0) = 1.0$</td>
</tr>
<tr>
<td>Position</td>
<td>$x$</td>
<td>$x'(t) = v$</td>
<td>$x(0.0) = 2.0$</td>
</tr>
</tbody>
</table>
Constant Acceleration -- Freefall

\[ F_y = m \frac{dv_y}{dt} \]

\[ -mg = m\dot{v}_y \]

\[ \dot{v}_y = -g \]

\[ \dot{y} = v_y \]
Air Resistance -- Terminal Velocity

\[ F_{net,y} = m \frac{dv_y}{dt} \]

\[ F_g + R = m \dot{v}_y \]

\[ mg - bv_y = m \dot{v}_y \]

\[ \dot{v}_y = \left( \frac{b}{m} \right)v_y - g \]

\[ \ddot{y} = v_y \]
Projectile Motion -- Air Resistance

\[ \vec{F}_{\text{net}} = m \frac{d\vec{v}}{dt} \]
\[ \vec{F}_g = m\dot{\vec{v}} \]
\[ [0, -mg] = m[\dot{v}_x, \dot{v}_y] \]

\[ \ddot{v}_y = -g \]
\[ \ddot{v}_x = 0 \]
\[ \dot{y} = v_y \]
\[ \dot{x} = v_x \]
Projectile Motion -- With Air Resistance

\[ \vec{F}_{net} = m \frac{d\vec{v}}{dt} \]

\[ \vec{F}_g + \vec{R} = m \vec{\dot{v}} \]

\[ [0,-mg] + [-bv_x,-bv_y] = m[\dot{v}_x,\dot{v}_y] \]

\[ \dot{v}_y = -(b/m)v_y - g \quad \quad \dot{y} = v_y \]

\[ \dot{v}_x = -(b/m)v_x \quad \quad \dot{x} = v_x \]
Damped Harmonic Motion

\[ F_{\text{net},x} = m \frac{dv_x}{dt} \]

\[ F_{\text{spring}} + F_{\text{resistance}} = m \dot{v}_x \]

\[ -kx - cv_x = m \dot{v}_x \]

\[ \dot{v}_x = -(k/m)x - (c/m)v_x \]

\[ \dot{x} = v_x \]

\( k = \text{Spring Constant} \)

\( c = \text{Damping Coefficient} \)
Damped Driven Harmonic Motion

\[ F_{\text{net},x} = m \frac{dv_x}{dt} \]

\[ F_{\text{spring}} + F_{\text{resistance}} + F_{\text{applied}} = m\dot{v}_x \]

\[ -kx - cv_x + F_o \cos(\omega t) = m\dot{v}_x \]

\[ \ddot{v}_x = -(k/m)x - (c/m)v_x + (F_o/m)\cos(\omega t) \]

\[ \dot{x} = v_x \]

\( k = \text{Spring Constant} \)

\( c = \text{Damping Coefficient} \)

\( F_o = \text{Driving Force Magnitude} \)

\( \omega = \text{Driving Force Frequency} \)
Response to an Impulse Driving Force

LabVIEW $G$ spike function

\[
spike(x) = \begin{cases} 
1 & 0 < x < 1 \\
0 & \text{otherwise}
\end{cases}
\]

\[
spike\left(\frac{t-t_o}{T}\right) = \begin{cases} 
1 & t_o < t < t_o + T \\
0 & \text{otherwise}
\end{cases}
\]

\[
F_{\text{applied}}(t) = F_o \cdot spike\left(\frac{t-t_o}{T}\right)
\]
Coupled Harmonic Oscillators

\[ F_{\text{net},1} = -kx_1 - c(x_1 - x_2) = m\ddot{x}_1 \]
\[ \dot{x}_1 = -(k/m)x_1 - (k/m)(x_1 - x_2) \]

\[ F_{\text{net},2} = -kx_2 - c(x_2 - x_1) = m\ddot{x}_2 \]
\[ \dot{x}_2 = -(k/m)x_2 - (k/m)(x_2 - x_1) \]
Charge in Magnetic Field

\[ \vec{B} = [0, 0, B] \]

\[ \vec{F}_{\text{net}} = m \frac{d\vec{v}}{dt} \]

\[ \vec{F}_{\text{Lorentz}} = m \dot{\vec{v}} \]

\[ q \ \vec{v} \times \vec{B} = m \dot{\vec{v}} \]

\[ \dot{\vec{v}} = \left( \frac{q}{m} \right) \vec{v} \times \vec{B} \]

\[ [\dot{v}_x, \dot{v}_y, \dot{v}_z] = \left( \frac{q}{m} \right) [v_y B, -v_x B, 0] \]

\[ \vec{v} \times \vec{B} = 0 \]

\[ \vec{v} = \frac{qB}{m} \]

\[ \dot{v}_x = \frac{qB}{m} v_y \]

\[ \dot{v}_y = -\frac{qB}{m} v_x \]

\[ \dot{v}_z = 0 \]

Figure 3.7 The path of a charged particle moving perpendicular to a uniform magnetic field.

Figure 3.8 Particle motion in a uniform magnetic field.